Supplementary exercises for sheet 6

You don't have to hand out those supplementary exercises; if you struggle with one of them, Alix Deleporte (who wrote them) will be happy to help!

Independence of real random variables

Exercise 1. Give a necessary and sufficient condition for a random variable X to be independent from itself.

Exercise 2. Let U and X be two independent random variables: U follows a uniform law on $[0,1]$ and X follows $\mathcal{E}(1)$.

1. Compute $\mathbb{P}(\max(U,X) \leq t)$ for every $t > 0$.
2. Draw the density of $\max(U,X)$.

Exercise 3. Let $(X_n)_{n\geq 1}$ be a sequence of real random variables on a probabilistic set $(\Omega, \mathbb{F}, \mathbb{P})$. Suppose that there exists a random variable X on $(\Omega, \mathbb{F}, \mathbb{P})$ such that $X_n \to X$ almost surely. (reminder: this means that for almost all ω, $X_n(\omega) \to X(\omega)$).

For every $\epsilon > 0$ and $n \in \mathbb{N}$, let

$$A_n^\epsilon = \{\omega \in \Omega, |X_n(\omega) - X(\omega)| < \epsilon\}.$$

Prove carefully that, for every $\epsilon > 0$, as $n \to +\infty$, $\mathbb{P}(A_n^\epsilon \to 1)$.

Exercise 4. Let $X_1, \ldots, X_n$ be independent, identically distributed (iid) real random variables, with density f. We are interested in the random variable

$$Z = \max(X_1, \ldots, X_n) - \min(X_1, \ldots, X_n).$$

We let $F : \mathbb{R} \to [0,1]$ be the following function: $F : t \mapsto \mathbb{P}(X_1 < t)$.

1. Show that, for all $u < v$,

   $$\mathbb{P}(\min(X_1, \ldots, X_n) \leq u, \max(X_1, \ldots, X_n) < v) = (F(v) - F(u))^n.$$

   Show that, for all $t > 0$,

   $$\mathbb{P}\left(\cap_{j=2}^n \{X_1 < X_j < X_1 + t\}\right) = \int_{\mathbb{R}} (F(x+t) - F(x))^{n-1} f(x) \, \mathrm{d}x.$$

   Show that, for all $t > 0$,

   $$\{Z < t\} = \cup_{i=1}^n \cap_{j \neq i} \{X_i < X_j < X_i + t\}$$

   and that the union is disjoint.

2. Deduce an expression of $t \mapsto \mathbb{P}(Z < t)$ involving f and F.

3. We now suppose that $X_1, \dots, X_n$ follow a uniform distribution on $[0,1]$. Make the above question explicit, show that Z admits a density, and determine this density.

Exercise 5. The Gamma function $\Gamma : (0,+\infty) \to (0,+\infty)$ is defined as follows: $\Gamma : a \mapsto \int_0^{+\infty} x^{a-1} e^{-x} \, \mathrm{d}x$.

Given $a > 0$ and $\lambda > 0$, the gamma distribution $\gamma_{a,\lambda}$ is defined by its density $f_{a,\lambda} : \mathbb{R} \to \mathbb{R}^+$ relatively to the Lebesgue measure:

$$f_{a,\lambda} : x \mapsto \frac{\lambda^a}{\Gamma(a)} \exp(-\lambda x) x^{a-1} \mathbb{1}(x \geq 0).$$

1. Show that this is a probability distribution; determine its expectation.

2. Let $V_1, \dots, V_n$ be iid random variables, with law $\mathcal{E}(\lambda)$. Show, by induction, that the law of $V_1 + V_2 + \ldots + V_n$ is $\gamma_{n,\lambda}$.

3. Let X and Y be two iid random variables with respective laws $\gamma_{a,\lambda}$ and $\gamma_{b,\lambda}$. Show that $X + Y$ follows the law $\gamma_{a+b,\lambda}$.

4. Let Z be a real random variable following $\mathcal{N}(0,1)$. Show that Z^2 follows $\gamma(\frac{1}{2}, \frac{1}{2})$. What is the distribution of the sum of n squares of iid standard Gaussian random variables?

Exercise 6. Let $(X_n)_{n \geq 1}$ a sequence of independent random variables, taking values in $[0,1]$. The goal of this exercise is to show that the following conditions are equivalent:

- (i) $\sum_{n=1}^{\infty} \mathbb{E}[X_n] < \infty$.
- (ii) almost surely, $\sum_{n=1}^{+\infty} X_n < \infty$.
- (iii) with non-zero probability, $\sum_{n=1}^{+\infty} X_n < \infty$.

For every n, we let $S_n = X_1 + \ldots + X_n$, and we let $S = \lim_{n \to +\infty} S_n \in [0,+\infty]$. In particular, condition (ii) can be written $\mathbb{P}(S < \infty) = 1$; clearly (ii) implies (iii).

1. Suppose (i) holds. Prove that $\mathbb{E}\left[\sum_{n=1}^{+\infty} X_n\right] < \infty$, then prove (ii).

   In the next questions we assume that (iii) holds.

2. Show that $\mathbb{E}[e^{-S}] > 0$.

3. Show that, for all $n \geq 1$,

$$\mathbb{E}[e^{-S_n}] = \prod_{k=1}^{n} \mathbb{E}[e^{-X_k}].$$

   Deduce (with a correct argument) an expression for $\mathbb{E}[e^{-S}]$.

4. Show that
$$-\sum_{n=1}^{\infty} \ln(\mathbb{E}[e^{-X_n}]) < \infty.$$

5. Show that, for all $x \in [0,1]$,
$$e^{-x} \leq 1 - e^{-1}x \leq e^{-e^{-1}x}.$$
(Hint: use convexity).

6. Deduce that
$$\mathbb{E}[e^{-X_n}] \leq e^{-e^{-1}\mathbb{E}[X_n]}.$$

7. Conclude: (iii) implies (i)

Strong convergence and Borel-Cantelli

Exercise 7. Give an example of sequence of events $(A_n)_{n\geq 1}$ in a probability space Ω such that $\sum_{n=1}^{+\infty} \mathbb{P}(A_n) = +\infty$ but $\mathbb{P}(\limsup(A_n)) = 0$. (In particular, the A_n's are not independent !)

Exercise 8. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and such that $\forall x, y, |f(x) - f(y)| \leq C|x - y|$. Let $X_n \to X$ a.s.

1. Show that $\mathbb{E}[f(X_n)] - \mathbb{E}[f(X)] \to 0$.
2. Let $\epsilon > 0$. Show that $\mathbb{P}(|f(X_n) - f(X)| \geq \epsilon) \to 0$.

Exercise 9. We want to generalise Exercise 6 of this sheet to the case when $X_1, X_2, \ldots$ take values in $[0, +\infty[$; we still want to characterise the fact that $\sum_{n=1}^{+\infty} X_n$ converges; again (ii) implies (iii).

1. Suppose (iii) again. Show that
$$\sum_{n=1}^{+\infty} \mathbb{P}(X_n > 1) < \infty.$$
(Hint: use Borel-Cantelli).

2. Suppose (iii). Show that
$$\sum_{n=1}^{\infty} \mathbb{E}[X_n \mathbb{1}_{X_n \leq 1}] < \infty.$$
(Use the previous exercise).

3. Suppose conversely that $\sum_{n=1}^{+\infty} \mathbb{P}(X_n > 1) < \infty$ and $\sum_{n=1}^{\infty} \mathbb{E}[X_n \mathbb{1}_{X_n \leq 1}] < \infty$. Show that, almost surely, there is a finite number of n's such that $X_n > 1$, and that $\sum_{n\geq 1} X_n \mathbb{1}_{X_n \leq 1} < \infty$. Conclude that (ii) is true.

Exercise 10. We want to improve on Exercise 7 in Sheet 6. Let $\lambda > 0$ and let $(X_n)_{n\geq 1}$ be a sequence of iid real random variables, with law $\mathcal{E}(\lambda)$. We already proved that the sequence

$$\frac{\max(X_1, \ldots, X_n)}{\ln(n)}$$

converges to $\frac{1}{\lambda}$ in probability. Let us prove that the convergence is, in fact, almost sure.

1. Prove that, for all $\epsilon > 0$, almost surely there exists only a finite number of n's such that
$$\frac{\max(X_1, \ldots, X_n)}{\ln(n)} \leq \frac{1}{\lambda} - \epsilon,$$
even though the sequence $(\frac{\max(X_1,\ldots,X_n)}{\ln(n)})_{n\geq 1}$ is not independent.

2. Show that the following statements are equivalent:
   - Almost surely, the number of n such that
   $$\frac{1}{\ln(n)} \max(X_1, \ldots, X_n) \geq \frac{1}{\lambda} + \epsilon$$
   is finite.
   - Almost surely, the number of n such that
   $$\frac{1}{\ln(n)} X_n \geq \frac{1}{\lambda} + \epsilon$$
   is finite.

3. Deduce that the sequence $(\frac{\max(X_1,\ldots,X_n)}{\ln(n)})_{n\geq 1}$ converges almost surely to $\frac{1}{\lambda}$.

Exercise 11. Let $(X_n)_{n\geq 1}$ be a sequence of iid, square-integrable real random variables, with expectation m and variance σ^2.

1. Show that for all $\delta > 0$, with probability at least $1 - \delta$ one has
$$\left| \frac{1}{n} \sum_{i=1}^{n} X_i - m \right| \leq \frac{\sigma}{\sqrt{\delta n}}.$$
Deduce that $\frac{1}{n} \sum_{i=1}^{n} X_i$ converges in probability to m.

2. Suppose now that X_n follow $\mathbb{N}(m, \sigma)$. Give a simple formula for the function $M : \mathbb{R} \to \mathbb{R}$ defined by

$$M : u \mapsto \mathbb{E}[\exp(u(X_1 - m))].$$

3. Prove that, for every $a \in \mathbb{R}$ and every $u > 0$,

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq m + \frac{a}{n}\right) \leq e^{-ua}(M(u))^n;$$

show a similar inequality for $\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \leq m - \frac{a}{n}\right)$.

4. Deduce that, for every $\epsilon > 0$,

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - m\right| \geq \epsilon\right) \leq 2\exp\left(\frac{-n\epsilon^2}{2\sigma^2}\right).$$

5. Find δ such that $\frac{\sigma}{\sqrt{\delta n}} = \epsilon$ and compare the above bound with the one in question 1.