Sheet 11

Please put your solution in the box of your corresponding instructor before **Friday, May 15th** at **12h00** and name your sheet "<First name>_<Last name>.pdf".

Exercise 1. We let $p \in (0,1)$, and, for every $n \in \mathbb{N}$, X_n be a random variable following a Binomial distribution with parameters (n,p).

1. For all $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$, give a simple formula for $\mathbb{E}[e^{\lambda X_n}]$. Prove that, for all $x \geq 0$,
$$\mathbb{P}[X_n \geq nx] \leq \exp\Big[n\Big(\log(1+p(e^{\lambda}-1)) - \lambda x\Big)\Big].$$

2. Suppose $1 > x > p$. Optimize the above inequality over λ (taking the necessary precautions) and find a function I such that, if $x > p$,
$$\mathbb{P}[X_n \geq nx] \leq \exp(nI(x)).$$
Study the function I; in particular, discuss the similarities and differences with the statement of Cramér's theorem.

3. If $x < p$, how does one study $\mathbb{P}[X_n \leq nx]$?

Exercise 2. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{N}$: for every n, μ_n is a sequence $(\mu_{n,k})_{k\in\mathbb{N}}$ of elements of $[0,1]$, such that $\sum_{k=1}^{+\infty} \mu_{n,k} = 1$ for every n.

1. Suppose that $(\mu_n)_{n\in\mathbb{N}}$ converges[1] to a probability measure $\overline{\mu}$. Show that
   a) For every $k \in \mathbb{N}$, as $n \to +\infty$, one has $\mu_{n,k} \to \overline{\mu}_k$.
   b) For every $\epsilon > 0$, there exists N such that, for every $n \in \mathbb{N}$,
$$\mu_n(\{1,\ldots,N\}) \geq 1-\epsilon.$$
Hint: first prove b) for n large enough (depending on ϵ but not on N), then increase N to get all $n \in \mathbb{N}$.

2. Suppose reciprocally that there exists a sequence $\overline{\mu}_k$ such that conditions a) and b) above are satisfied. Prove that $\overline{\mu}$ is a probability measure and that $(\mu_n)_{n\in\mathbb{N}}$ converges to μ.

3. Suppose that condition b) above is satisfied. Show that there exists a probability sequence $\overline{\mu}$ such that you can extract a subsequence of $(\mu_n)_{n\in\mathbb{N}}$ converging towards $\overline{\mu}$. (You may use a diagonal argument).

Exercise 3.

[1] Here, a bounded continous function f from $\mathbb{N}$ to $\mathbb{R}$ simply means a bounded real sequence; compact support means finite number of non-zero elements.

1. Using exercise sheet 9, prove rapidly that, if $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ are sequence of real random variables such that $(X_n)_{n\in\mathbb{N}}$ converges in distribution to some random variable X and $(Y_n)_{n\in\mathbb{N}}$ converges (in distribution/probability) to a constant c, then
$$X_n + Y_n \Rightarrow X + c \qquad\qquad X_n Y_n \Rightarrow cX.$$

2. We let $(X_i)_{i\in\mathbb{N}}$ be a sequence of independent, identically distributed Gaussian variables of expectation μ and variance σ. We consider their *empirical* variance and mean:
$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad\qquad V_n(X) = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2.$$
What are the expectations of $\overline{X}_n$ and $V_n(X)$? Show that, as $n \to +\infty$, these random variables converge (in an appropriate sense) to their expectations.

3. Show that, as $n \to +\infty$,
$$\frac{X_1 + \ldots + X_n - n\mu}{\sqrt{nV_n(X)}} \Rightarrow \mathcal{N}(0,1).$$
Can one replace μ with $\overline{X}_n$ in the result above?

Exercise 4. Return to Sheet 4, exercise 2: let μ be a probability measure on $\mathbb{N}$, described as usual by a sequence $(\mu_k)_{k\in\mathbb{N}}$. A virus spreads among the population; at time t, a number $N(t)$ of viruses are alive; between times t and $t+1$, each virus dies and creates new viruses in a random, independent and identical way: each virus creates k new viruses with probability μ_k. At time zero, we start with a deterministic number $N(0)$ of viruses.

Recall that, in sheet 4, we assumed that $R_0 := \sum_{k=0}^{+\infty} k\mu_k < 1$ and concluded that almost surely, the epidemic disappears in finite time. We suppose now that $R_0 = 1$ and $\mu_0 \neq 0$.

1. Show that the expected number of viruses is constant in time.

2. We denote by X an integer-valued random variable following μ. Using Jensen's inequality, prove that, for every $\alpha \in [0,1)$, one has $\mathbb{E}[\alpha^X] > \alpha$. Deduce that, for any $\alpha_0 \in [0,1]$, the sequence $(\alpha_n)_{n\in\mathbb{N}}$ defined by $\alpha_{n+1} = \mathbb{E}[\alpha_n^X]$ converges to 1.

3. We let $\alpha_0 = \mu_0$. Prove that
$$\mathbb{P}[N(t+1) = 0] = \mathbb{E}[\mu_0^{N(t)}] = \mathbb{E}[\alpha_1^{N(t-1)}].$$
Continue by induction, and show that $N(t)$ converges in probability to zero as $t \to +\infty$. Does one have almost sure convergence? Convergence in L^1?