Sheet 6

Please email your solution to your corresponding instructor before **Friday, April 3rd** at **12h00** by an email with object “<First name> <Last name> Stochastik Sheet 6” and an attachment “<First name>_<Last name>.pdf”.

In this exercise sheet we use the following notations.
For $\lambda > 0$, the exponential law $\mathcal{E}(\lambda)$ with parameter λ is the law of a real random variable with the following density with respect to the Lebesgue measure:

$$\mathrm{d}\mathbb{P}_{\mathcal{E}(\lambda)} = \mathbb{1}(x \geq 0)\lambda e^{-\lambda x}\,\mathrm{d}x.$$

For $\mu \in \mathbb{R}$ and $\sigma > 0$, the Gaussian law $\mathcal{N}(\mu, \sigma)$ with mean μ and variance σ^2 is the law of a real random variable with the following density with respect to the Lebesgue measure:

$$\mathrm{d}\mathbb{P}_{\mathcal{N}(\mu,\sigma)} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,\mathrm{d}x.$$

We also write “iid” for “independent, identically distributed”.
Exercise 1. Let U, V be two independent random variables of law $\mathcal{E}(1)$.

1. What are the laws of $\max(U, V)$ and $\min(U, V)$?

   Indication: consider $\mathbb{P}(\max(U, V) \leq t)$ and $\mathbb{P}(\min(U, V) \geq t)$.

2. What is the law of $U + V$?

   Indication: compute the density.

Solution of exercise 1.

1. The indication is a standard trick when considering the max or min of two random variables. Use it often!

   By independence, one has

   $$\mathbb{P}(\max(U, V) \leq t) = \mathbb{P}(\{U \leq t\} \cap \{V \leq t\}) = \mathbb{P}(U \leq t)\mathbb{P}(V \leq t) = (1 - e^{-\lambda t})^2.$$

   In particular, $\max(U, V)$ has the density $t \mapsto \frac{\mathrm{d}}{\mathrm{d}t}(1 - e^{-\lambda t})^2 = 2\lambda(e^{-\lambda t} - e^{-2\lambda t})$.

   In the same way, one has

   $$\mathbb{P}(\min(U, V) \geq t) = (e^{\lambda t})^2,$$

   so that $\min(U, V)$ has density $2\lambda e^{-2\lambda t}$: $\min(U, V)$ follows $\mathcal{E}(2)$.

2. It is a general fact that the density of $U+V$ is the convolution of the densities of U and V. Indeed, in general if the density of U is given by $f:\mathbb{R}\to\mathbb{R}$, and the density of V is given by $g:\mathbb{R}\to\mathbb{R}$, then, by Fubini and a change of variables,

$$\begin{aligned}\mathbb{P}(U+V\leq t) &= \int_{-\infty}^{+\infty} f(x)\left(\int_{-\infty}^{t-x} g(y)\,\mathrm{d}y\right)\mathrm{d}x\\ &= \int_{-\infty}^{+\infty} f(x)\left(\int_{-\infty}^{t} g(y-x)\,\mathrm{d}y\right)\mathrm{d}x\\ &= \int_{-\infty}^{t}\left(\int_{-\infty}^{+\infty} g(y-x)f(x)\,\mathrm{d}x\right)\mathrm{d}y.\end{aligned}$$

Hence, if we let

$$h: y\mapsto \int_{-\infty}^{+\infty} g(y-x)f(x)\,\mathrm{d}x,$$

we have, for every $t\in\mathbb{R}$,

$$\mathbb{P}(U+V\leq t) = \int_{-\infty}^{t} h(y)\,\mathrm{d}y,$$

so that $U+V$ has density $h(y)\,\mathrm{d}y$.

Here, for every $y\in\mathbb{R}$,

$$h(y) = \mathbb{1}_{y\geq 0}\int_0^y e^{-(y-x)}e^{-x}\,\mathrm{d}x = ye^{-y}.$$

□

Exercise 2. Let X and Y be two independent random variables of law $\mathcal{N}(0,1)$.

1. Show that $X+Y$ and $X-Y$ are Gaussian random variables; what are their means and variances?

2. Show that $X+Y$ and $X-Y$ are independent.

Solution of exercise 2. The joint density of X,Y is

$$(x,y)\mapsto \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)} = \frac{1}{2\pi}e^{-\frac{1}{4}((x+y)^2+(x-y)^2)}.$$

In particular, the joint density of $(U,V)=(X+Y,X-Y)$ is

$$(u,v)\mapsto \frac{1}{4\pi}e^{-\frac{1}{4}(u^2+v^2)} = \left(\frac{1}{\sqrt{2}\sqrt{2\pi}}e^{-\frac{1}{4}u^2}\right)\left(\frac{1}{\sqrt{2}\sqrt{2\pi}}e^{-\frac{1}{4}v^2}\right)$$

since we remembered to take into account the Jacobian when changing variables.

In particular, $X+Y$ and $X-Y$ are independent, each of them follows $\mathcal{N}(0,\sqrt{2})$. □

Exercise 3. Marco and Hairuo want to talk with each other about the Stochastik course. They agreed to do a video-conference talk at some point between 10 AM and 11 AM. Each of them arrives at a random time, following a uniform distribution between 10 AM and 11 AM, independently of each other. They wait 15 minutes for the other person, they leave if the other person does not arrive.

What is the probability that they will talk to each other? (Indication: draw a picture).

Solution of exercise 3. The joint distribution of the time at which Marco and Hairuo arrive corresponds to the uniform distribution on the unit square in $\mathbb{R}^2$. The event "The interval between their arrival times is smaller than 15 minutes" has for complement the union of the triangle with coordinates $(0, \frac{1}{4}), (\frac{3}{4}, 1), (0, 1)$ and the triangle with coordinates $(\frac{1}{4}, 0), (1, \frac{3}{4}), (1, 0)$. This yields

$$\mathbb{P}[\text{they meet}] = \frac{7}{16}.$$

□

Exercise 4. One throws two dice and denote by X and Y, respectively, the greater and the smaller of the two numbers obtained. Are the random variables X and Y independent?

Solution of exercise 4. For instance, one has

$$\mathbb{P}[X = 1] = \frac{1}{36} \qquad \mathbb{P}[Y = 1] = \frac{11}{36} \qquad \mathbb{P}(X = 1, Y = 1) = \frac{1}{36}.$$

In particular, $\mathbb{P}(X = 1, Y = 1) \neq \mathbb{P}[Y = 1]\mathbb{P}[X = 1]$ so that X and Y are not independent.
□

Exercise 5. One can collect plastic dinosaurs when buying breakfast cereals. There are n different dinosaurs to collect, and each cereal package contains one of them (at random, uniformly and independently).

In average, how many packages does one have to buy to collect all n dinosaurs?

Solution of exercise 5. By linearity of expectation, this problem can be solved by a summation over the following sub-problem: if one has already collected j dinosaurs, how many boxes in average does one have to buy to get another one?

As long as one has j dinosaurs, each new package brings a new dinosaur with probability $p = \frac{n-j}{n} = 1 - \frac{j}{n}$. Using a probability tree, or, if you want to be really formal, adapting Sheet 4, Exercise 5, Question 3 to a sequence of iid Bernoulli flips, the probability that the first new dinosaur appears at the k-th box is $p(1-p)^{k-1}$. In particular, the average number of boxes is

$$\sum_{k=1}^{+\infty} kp(1-p)^{k-1} = \frac{1}{p} = \frac{n}{n-j}.$$

Summing over j yields that the average number of boxes is

$$\sum_{j=0}^{n-1} \frac{n}{n-j} = n \sum_{k=1}^{n} \frac{1}{k}.$$

Note that, as $n \to +\infty$, the number of boxes to be opened is equivalent to $n \log(n)$. $\square$

Exercise 6. Let $(p_n)_{n\geq 1}$ a sequence with values in $[0,1]$, tending to zero as $n \to +\infty$. Let (X_n) a sequence of independent random variables; for every n, the law of X_n is a Bernoulli with parameter p_n.

1. Show that $X_n \to 0$ in probability.
2. Under what condition on $\sum p_n$ does the sequence (X_n) converge almost surely to 0?

Solution of exercise 6.

1. By definition, for all $\epsilon > 0$,

   $$\mathbb{P}[|X_n| > \epsilon] = \mathbb{P}[X_n > \epsilon] = p_n.$$

   Since p_n tends to zero, X_n converges to 0 in probability.

2. This is a direct application of Borel-Cantelli: if

   $$\sum p_n = \infty,$$

   then almost surely there exists an infinite number of n's such that $X_n = 1$; in particular, (X_n) does not converge almost surely to zero.

   Conversely, if

   $$\sum p_n < \infty,$$

   then almost surely there exists only a finite number of X_n's such that $X_n = 1$, so that (X_n) converges almost surely to 0.

$\square$

Exercise 7. Let $\lambda > 0$ and let $(X_n)_{n\geq 1}$ be a sequence of iid real random variables, with law $\mathcal{E}(\lambda)$.

1. Show that, for all $t > 1$, as $n \to +\infty$,

   $$\mathbb{P}\left(\max(X_1, \dots, X_n) \leq \frac{t \log(n)}{\lambda}\right) \to 1.$$

2. Show that, for all $0 < t < 1$, as $n \to +\infty$,

   $$\mathbb{P}\left(\max(X_1, \dots, X_n) \leq \frac{t \log(n)}{\lambda}\right) \to 0.$$

3. Deduce that $\frac{1}{\log(n)}\max(X_1,\dots,X_n)$ converges in probability to $\frac{1}{\lambda}$.
4. Bonus: show almost sure convergence (follow exercise 10 of the supplementary sheet).

Solution of exercise 7.

1. As in Exercise 1, one has
$$\mathbb{P}\left(\max(X_1,\dots,X_n) \le \frac{t\log(n)}{\lambda}\right) = (1-e^{-t\log(n)})^n = (1-n^{-t})^n.$$
If $t>0$, then $\log(1-n^{-t}) \sim -n^{-t}$, so that, if $t>1$,
$$(1-n^{-t})^n = \exp(n\log(1-n^{-t})) \to 1.$$
2. If $0<t<1$, then $(1-n^{-t})^n \to 0$.
3. By the two previous questions, for $\epsilon>0$,
$$\mathbb{P}\left[\frac{1}{\log(n)}\max(X_1,\dots,X_n) - \frac{1}{\lambda} > \epsilon\right] = \mathbb{P}\left[\max(X_1,\dots,X_n) \le \frac{\log(n)(1+\lambda\epsilon)}{\lambda}\right] \to 0,$$
and similarly
$$\mathbb{P}\left[\frac{1}{\log(n)}\max(X_1,\dots,X_n) - \frac{1}{\lambda} < -\epsilon\right] = \mathbb{P}\left[\max(X_1,\dots,X_n) \ge \frac{\log(n)(1-\lambda\epsilon)}{\lambda}\right] \to 0,$$
hence the claim.

Here is the solution of the bonus question:

1. For all $\epsilon>0$, one has
$$\mathbb{P}\left[\max(X_1,\dots,X_n) \le \left(\frac{1}{\lambda}-\epsilon\right)\log(n)\right] = (1-n^{-1+\lambda\epsilon})^n = \exp(n\log(1-n^{-1+\lambda\epsilon})) \le \exp(-n^{\lambda\epsilon}),$$
where we used the inequality $\log(1-x) \le -x$ valid for all $x<1$.
Since the series $(\exp(-n^{\lambda\epsilon}))_{n\ge1}$ is summable, one can apply the first part of Borel-Cantelli: almost surely, there exists only a finite number of n's such that
$$\max(X_1,\dots,X_n) \le \left(\frac{1}{\lambda}-\epsilon\right)\log(n).$$
2. Clearly the first statement implies the second one; reciprocally, since $\log(n)$ tends to infinity with n, if for all n larger than n_0, one has $\frac{1}{\log(n)}X_n \le \frac{1}{\lambda}+\epsilon$, then for all $n \ge n_0$ one has
$$\frac{1}{\log(n)}\max(X_1,\dots,X_n) \ge \max\left(\frac{1}{\lambda}+\epsilon, \frac{1}{\log(n)}\max(X_1,\dots,X_{n_0})\right).$$
In particular, there exists n_1 such that, for all $n \ge n_1$, one has
$$\frac{1}{\log(n)}\max(X_1,\dots,X_n) \ge \frac{1}{\lambda}+\epsilon.$$

3. Making question 1 more explicit, one has

$$\mathbb{P}\left[\frac{1}{\log(n)}\max(X_1,\ldots,X_n) - \frac{1}{\lambda} > \epsilon\right] \sim n^{-\lambda\epsilon}.$$

For ϵ small enough, the series $n^{-\lambda\epsilon}$ is not summable. The problem is that one cannot apply directly the second part of Borel-Cantelli because $(\max(X_1,\ldots,X_n))_n$ is not a sequence of independent random variables.

Let us check the second statement of the previous question using the Borel-Cantelli lemma: there holds

$$\mathbb{P}\left[\frac{X_n}{\log(n)} \geq \frac{1}{\lambda} + \epsilon\right] = e^{-\lambda\left(\frac{1}{\lambda}+\epsilon\right)\log(n)} = n^{-1-\epsilon}.$$

For all $\epsilon > 0$, the series $(n^{-1-\epsilon})_{n\geq 1}$ is summable, hence the second statement is true. Hence, for all $\epsilon > 0$, almost surely, as $n \to +\infty$ one has

$$\frac{\max(X_1,\ldots,X_n)}{\log(n)} \leq \frac{1}{\lambda} + \epsilon.$$

This concludes the proof of almost sure convergence.

□