Sheet 6

Please email your solution to your corresponding instructor before **Friday, April 3rd** at **12h00** by an email with object “<First name> <Last name> Stochastik Sheet 6” and an attachment “<First name>_<Last name>.pdf”.

In this exercise sheet we use the following notations.

For $\lambda > 0$, the exponential law $\mathcal{E}(\lambda)$ with parameter λ is the law of a real random variable with the following density with respect to the Lebesgue measure:

$$\mathrm{d}\mathbb{P}_{\mathcal{E}(\lambda)} = \mathbb{1}(x \geq 0)\lambda e^{-\lambda x}\,\mathrm{d}x.$$

For $\mu \in \mathbb{R}$ and $\sigma > 0$, the Gaussian law $\mathcal{N}(\mu,\sigma)$ with mean μ and variance σ^2 is the law of a real random variable with the following density with respect to the Lebesgue measure:

$$\mathrm{d}\mathbb{P}_{\mathcal{N}(\mu,\sigma)} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,\mathrm{d}x.$$

We also write “iid” for “independent, identically distributed”.

Exercise 1. Let U, V be two independent random variables of law $\mathcal{E}(1)$.

1. What are the laws of $\max(U,V)$ and $\min(U,V)$?

   Indication: consider $\mathbb{P}(\max(U,V) \leq t)$ and $\mathbb{P}(\min(U,V) \geq t)$.

2. What is the law of $U+V$?

   Indication: compute the density.

Exercise 2. Let X and Y be two independent random variables of law $\mathcal{N}(0,1)$.

1. Show that $X+Y$ and $X-Y$ are Gaussian random variables; what are their means and variances?

2. Show that $X+Y$ and $X-Y$ are independent.

Exercise 3. Marco and Hairuo want to talk with each other about the Stochastik course. They agreed to do a video-conference talk at some point between 10 AM and 11 AM. Each of them arrives at a random time, following a uniform distribution between 10 AM and 11 AM, independently of each other. They wait 15 minutes for the other person, they leave if the other person does not arrive.

What is the probability that they will talk to each other? (Indication: draw a picture).

Exercise 4. One throws two dice and denote by X and Y, respectively, the greater and the smaller of the two numbers obtained. Are the random variables X and Y independent?

Exercise 5. One can collect plastic dinosaurs when buying breakfast cereals. There are n different dinosaurs to collect, and each cereal package contains one of them (at random, uniformly and independently).

In average, how many packages does one have to buy to collect all n dinosaurs?

Exercise 6. Let $(p_n)_{n\geq 1}$ a sequence with values in $[0,1]$, tending to zero as $n \to +\infty$. Let (X_n) a sequence of independent random variables; for every n, the law of X_n is a Bernoulli with parameter p_n.

1. Show that $X_n \to 0$ in probability.
2. Under what condition on $\sum p_n$ does the sequence (X_n) converge almost surely to 0?

Exercise 7. Let $\lambda > 0$ and let $(X_n)_{n\geq 1}$ be a sequence of iid real random variables, with law $\mathcal{E}(\lambda)$.

1. Show that, for all $t > 1$, as $n \to +\infty$,
$$\mathbb{P}\left(\max(X_1, \ldots, X_n) \leq \frac{t\ln(n)}{\lambda}\right) \to 1.$$
2. Show that, for all $0 < t < 1$, as $n \to +\infty$,
$$\mathbb{P}\left(\max(X_1, \ldots, X_n) \leq \frac{t\ln(n)}{\lambda}\right) \to 0.$$
3. Deduce that $\frac{1}{\ln(n)}\max(X_1, \ldots, X_n)$ converges in probability to $\frac{1}{\lambda}$.
4. Bonus: show almost sure convergence (follow exercise 10 of the supplementary sheet).