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ON THE HEAT KERNEL OF A COMPLETE RIEMANNIAN MANIFOLD

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Let L be a second order elliptic operator without zero order term on a manifold. Then K. Ito [6] has constructed a diffusion of M governed by L. From this diffusion we have a system of probability measures on M which also give rise to a semi-group T_t defined on the space of bounded measurable functions on M.

As usual, one restricts the semi-group T_t to B, the space of bounded measurable functions u with the property that $\lim_{t\to 0} T_t u(x) = u(x)$ for all $x \in M$. (This includes the space of bounded continuous functions.) Then the generator A of T_t restricted to B can be defined as follows. The domain of A is the space of functions u in B so that $1/t\,(T_t u(x) - u(x))$ converges boundedly to a function in B when t tends to zero. For any function u in the domain of A, $A u(x)$ is defined to be

$$\lim_{t\to 0} \frac{1}{t}(T_t u(x) - u(x)).$$

It is known that $A u = L u$ when u is a smooth function with compact support. However, if we merely assume u is an element in the domain of A or the domain of L, the equality $A u = L u$ need not be true. In order to fully utilize the semi-group theory, it is desirable to find a condition under which $A = L$. In the first part of this paper, we prove that if L is the Laplacian of a complete Riemannian manifold with Ricci curvature bounded from below, then indeed $A = L$. In fact, we give a complete (different from Ito's) self-contained construction of the probability measures by generalizing the classical Hille-Yosida theory. As a result, the smoothness of the coefficient of L is weaker than those in Ito's approach. Furthermore, we find that if we restrict T_t to the closure of the domain of L (with respect to the supremium norm), then convergence in many cases can be taken in the strong sense.

On the second part of the paper, we show that the kernels cannot be too large at infinity for a complete manifold whose Ricci curvature is bounded from below. Precisely, we show that the semi-group generated by the Laplacien must preserve functions which vanish at infinity.

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1. The construction of the probability measures

Let L be a second order elliptic differential operator defined on a n-dimensional differentiable manifold M such that in local coordinates, L can be expressed as

$$\mathrm{L} = \sum_{i,j} a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_j b^j(x)\frac{\partial}{\partial x^j} + c(x), \tag{1.1}$$

where $(a^{ij}(x))$ is a positive definite symmetric matrix.

In order to make the Hille-Hosida theory applicable in our situation, we postulate the following hypothesis on L.

HYPOTHESIS A. — $a^{ij}(x)$, $b^j(x)$ are locally Hölder continuous and $c(x) = 0$.

HYPOTHESIS A′. — L can be written in the divergence form

$$\sum_{i,j}\frac{\partial}{\partial x^i}\left(a^{ij}(x)\frac{\partial}{\partial x^j}\right),$$

where $(a^{ij}(x))$ is a positive definite matrix and $a^{ij}(x)$ are locally bounded measurable functions.

HYPOTHESIS B. — There is a non-negative locally Lipschitz function r defined on M so that L (r), in the distribution sense, is bounded from above on M and r tends to infinity uniformly when x tends to infinity.

HYPOTHESIS C. — In hypothesis B, we assume

$$\lim_{a\to\infty} a^{-2}\sup_{r\leqq a}\{|\nabla r|^2 = \sum_{i,j} a^{ij} r_i r_j\} = 0.$$

Under either hypothesis A, B, C or A′, B, C; we shall construct a semi-group associated to L. Since we shall apply the theory to Laplacian on a Riemannian manifold, we shall only provide proofs under hypothesis A′, B and C. (The proof for the other case follows by maximum principle.) In this case, r will be the distance from a fixed point so that hypothesis C is automatic while hypothesis B follows from the assumption that the Ricci curvature of M is bounded from below by a constant. One can also verify that these hypotheses are valid for complete submanifolds of euclidean space with bounded mean curvature.

Let B (M) be the space of all bounded Borel measurable functions defined on M so that two functions are identified if they differ by a set of measure zero. Naturally, B (M) is a Banach space equipped with the essential supremium norm. Our elliptic operator can be considered as an unbounded operator defined on B (M) whose domain is the space of functions f in B (M) so that Lf, in the distribution sense, is bounded.

PROPOSITION 1.1. — *Under hypothesis* A′ *and* B, $(\mathrm{L}-\lambda\,\mathrm{I})^{-1}$ *is a bounded operator defined on* B (M) *with norm not greater than* $1/\lambda$ *when* $\lambda > 0$.

Proof. — First of all, we note that the theory of DeGiorgi-Nash-Moser [5] can be applied to prove the following: if $f \in \mathrm{B}(\mathrm{M})$ and if $u \in \mathrm{B}(\mathrm{M})$ with locally finite Dirichlet integral so that $(\mathrm{L}-\lambda)\, u = f$ in the sense of distribution, then u is locally Hölder continuous. Furthermore, given compact domains K_i with smooth boundaries in M, we can find a continuous solution u_i of $(\mathrm{L}-\lambda)\, u = f$ so that $u_i \mid \partial \mathrm{K}_i = 0$ and u_i has locally finite Dirichlet integral.

We shall require the following:

LEMMA 1.1. — *Under hypothesis* A′ *and* B, *if* f, $u \in \mathrm{B}(\mathrm{M})$ *so that u is continuous with locally finite Dirichlet integral and* $(\mathrm{L}-\lambda)\, u = f$, *then* $\| u \| \leqq (1/\lambda) \| f \|$ *when* $\lambda > 0$.

Proof. — Let $\varepsilon > 0$ be an arbitrary number. Then for all constant $k > (1/\lambda) \| f \|$, the set

$$\mathrm{D}_\varepsilon = \{ x \mid u(x) - \varepsilon\, r(x) - k \geqq 0 \}$$

is compact.

Let $v = \max(u - \varepsilon\, r - k, 0)$. Then the Dirichlet integral of v is finite and v has compact support. Hence, from the equation $\mathrm{L}(u - \varepsilon\, r) = \lambda\, u - \varepsilon\, \mathrm{L}\, r + f$, we have

$$-\int_{\mathrm{D}_\varepsilon} \sum_{i,j} a^{ij}(x) \frac{\partial}{\partial x^i}(u - \varepsilon\, r) \frac{\partial}{\partial x^j}(u - \varepsilon\, r) = \int_{\mathrm{D}_\varepsilon} (\lambda\, u - \varepsilon\, \mathrm{L}\, r + f)(u - \varepsilon\, r - k). \tag{1.2}$$

Since $u(x) \geqq \varepsilon\, r(x) + k > \varepsilon\, r(x) + (1/\lambda) \| f \|$ on D_ε, we can choose ε small so that $\lambda\, u(x) - \varepsilon\, \mathrm{L}\, r(x) + f(x) > 0$ on D_ε. The equality (1.10) then shows that the set where $u(x) > \varepsilon\, r(x) + k$ has measure zero. Therefore,

$$u(x) \leqq \varepsilon\, r(x) + k \tag{1.3}$$

on M.

Let $\varepsilon \to 0$ and then $k \to (1/\lambda) \| f \|$ in (1.3), we have

$$\sup u(x) \leqq \frac{1}{\lambda} \| f \|. \tag{1.4}$$

By symmetry, we have $\| u \| \leqq (1/\lambda) \| f \|$ and lemma 1.1 is proved.

In order to prove proposition 1.1, it remains to solve $(\mathrm{L}-\lambda)\, u = f$ for arbitrary $f \in \mathrm{B}(\mathrm{M})$. Clearly, we may assume f is non-negative.

Let K_i be an increasing sequence of compact domains with smooth boundaries. Let u_i be the continuous solution of $(\mathrm{L}-\lambda)\, u = f$ on K_i so that $u \mid \partial \mathrm{K}_i = 0$. Since f is non-negative and $u_i \mid \partial \mathrm{K}_i = 0$, one can use the similar argument as above to prove that u_i is non-positive,

$$\inf_{x \in \mathrm{K}_i} u_i(x) \geqq \frac{-1}{\lambda} \| f \|$$

and u_i is monotonic decreasing for i tends to infinity. By the DeGiorgi-Nash-Moser theory, we have a local Hölder estimate of u_i so that $\lim\limits_{i \to \infty} u_i$ is also Hölder continuous.

On the other hand, it is easy to find a local estimate of the Dirichlet integral of u_i so that $\lim_{i\to\infty} u_i$ has locally finite Dirichlet integral and that $\lim_{i\to\infty} u_i$ is a solution of $(L-\lambda)u=f$ globally.

In order to apply the Hille-Yosida theory to L, one has to know that the domain of L is dense in B (M). Since this need not be the case, we consider V, the space of functions $u \in B(M)$ so that both Lu and $L^2 u$ are bounded in the distribution sense. Let E be the closure of the domain of L in B (M). Then we claim that V is dense in E.

In fact, let

$$J_n=\left(I-\frac{1}{n}L\right)^{-1}.$$

Then for u belongs to the domain of L,

$$LJ_n u = J_n L u = n(J_n - I)u \tag{1.5}$$

and so

$$(J_n u - u) = \frac{1}{n} J_n(Lu). \tag{1.6}$$

By proposition 1.1, the operators J_n has norm not greater than one. Therefore, $\lim_{n\to\infty} J_n u = u$ in B (M). As equation (1.5) shows $J_n u \in V$, our claim is proved.

Define a new operator $\bar{L}$ on E so that the domain of $\bar{L}$ is V and $\bar{L} = L \mid V$. Then $\bar{L}$ is a densely defined operator on E. As $(\bar{L}-\lambda I)^{-1} E \subset E$ for all $\lambda > 0$, we can apply the basic theorem [7] in Hille-Yosida theory to conclude that $\bar{L}$ is the infinitesimal generator of an equicontinuous semi-group of class (C_0). This means that we can find a one parameter family of bounded linear operators $\{T_t \mid t \geqq 0\}$ on E so that

$$T_t T_s = T_{t+s}, \tag{1.7}$$

$$T_0 = I, \tag{1.8}$$

$$\lim_{t\to t_0} T_t x = T_{t_0} x \tag{1.9}$$

for $t_0 \geqq 0$, $x \in E$.

Furthermore, the domain of $\bar{L}$ is exactly those elements $x \in E$ so that $\lim_{i\to\infty} h^{-1}(T_h - I)x$ exists and $\bar{L}x$ is given by this limit.

From the proof of proposition 1.1, we know that the (unique) solution $u \in B(M)$ of $(L-\lambda)u=f$ is non-positive when f is bounded and non-negative. Hence, we can check that T_t maps non-negative functions to non-negative functions. (At this point, one may use Riesz representation to show that T_t is represented by measures. But these measures, *a priori*, only represent T_t on functions which vanish at infinity. Therefore, we have to work harder.)

In order to extend T_t to the whole space $B(M)$, we have to use hypothesis C. Without loss of generality, we shall assume $\inf_{x \in M} r(x) = 0$. Let σ be a non-negative smooth function defined on the real line so that $\sigma'(t) \leqq 0$ for $t \geqq 0$, σ has support in $[-1, 1]$ and $\sigma(x) = 1$ for $|x| \leqq 1/2$. Then $r_a = \sigma(r/a)$ is a locally Lipschitz function defined on M such that $r_a(x) = 1$ for $r(x) \leqq a/2$, $r_a(x) = 0$ for $r(x) \geqq a$. Furthermore, $L r_a = (\sigma'/a) L r + (\sigma''/a^2) |\nabla r|^2$, in the distribution sense, is not less than, $-C_a = -a^{-1} A - a^{-2} B \sup_{r(x) \leqq a} |\nabla r|^2$ where A, B are positive constants depending on σ and the upper bound of Lr.

We claim that for all $k \geqq 0$,

$$J_n^k r_a \geqq r_a - k n^{-1} C_a, \tag{1.10}$$

$$L J_n^k r_a \geqq -C_a. \tag{1.11}$$

We assume both (1.10) and (1.11) are true for k. Then as $J_n^{k+1} r_a - (1/n) L J_n^{k+1} r_a = J_n^k r_a$ it suffices to prove

$$J_n^{k+1} r_a - J_n^k r_a \geqq -n^{-1} C_a. \tag{1.12}$$

Indeed, by (1.11), we have

$$L(J_n^{k+1} r_a - J_n^k r_a + \varepsilon r) \leqq n(J_n^{k+1} r_a - J_n^k r_a) + C_a + \varepsilon L r. \tag{1.13}$$

We multiply (1.13) by the function $\max(b - \varepsilon r - J_n^{k+1} r_a + J_n^k r_a, 0)$ with $b < -n^{-1} C_a$ and $\varepsilon > 0$. Then we integrate by parts and prove that when ε is small

$$J_n^{k+1} r_a - J_n^k r_a \geqq b - \varepsilon r. \tag{1.14}$$

Letting ε tend to zero, and b tend to $-n^{-1} C_a$ we obtain (1.12). Now consider the following operator

$$T_t^n = \exp(t L J_n) = \exp tn(J_n - 1) = \exp(-nt) \exp(nt J_n). \tag{1.15}$$

If x belongs to our Banach space E, then we know from classical theory that $\lim_{n \to \infty} T_t^n x = T_t x$ in the strong sense.

Since J_n is a bounded operator defined on $B(M)$ with norm not greater than one, the operator T_t^n is a bounded operator defined on $B(M)$ with norm not greater than one.

Let $\beta(M)$ be the Stone-Čech compactification of M so that the space of continuous functions on $\beta(M)$ is the same as the space of bounded continuous on M. Then as T_t^n is a non-negative operator, we can apply the Riesz representation theorem to find Borel measures $P_t^n(x, \mu)$ on $\beta(M)$ for each t and x such that

$$T_t^n u(x) = \int_{\beta(M)} p_t^n(x, \mu) u, \tag{1.16}$$

whenever u is continuous. (Note that by regularity theory in elliptic equation, $J_n u$ and hence $T_t^n u$ is always continuous).

We claim that the measures $p_t^n(x, \mu)$ are in fact supported in M.

Let r_a be the function defined above.

Then $\| r_a \| = 1$. For all $\varepsilon > 0$, let k be so large that

(1.17)
$$e^{-nt} \sum_{i=k+1}^{\infty} (nt)^k (k!)^{-1} \leqq \varepsilon.$$

Then:

1.18)
$$\begin{aligned} T_t^n r_a &= e^{-nt} \exp(nt\,J_n)\, r_a \\ &\geqq e^{-nt} \sum_{i=0}^{k} (nt)^k (k!)^{-1} J_n^k r_a - e^{-nt} \sum_{i=k+1}^{\infty} (nt)^k (k!)^{-1} \| r_a \| \\ &\geqq e^{-nt} \sum_{i=0}^{k} (nt)^k (k!)^{-1} (r_a - kn^{-1} C_a) - \varepsilon \geqq r_a - \varepsilon r_a - (1-\varepsilon)\, kn^{-1} C_a - \varepsilon. \end{aligned}$$

When a tends to infinity and ε tends to zero, (1.18) and hypotheses B and C show that

(1.19)
$$\lim_{a \to \infty} T_t^n r_a \geqq 1.$$

Putting (1.19) into (1.16), we see

(1.20)
$$\int_M p_t(x, \mu) \geqq 1.$$

Since $T_t^n 1 = 1$, $p_t^n(x, \mu)$ are probability measures so that (1.20) implies the support of $p_t^n(x, \mu)$ are in M. Hence, we can write:

(1.21)
$$T_t^n u(x) = \int_M p_t^n(x, \mu)\, u$$

when u is a bounded continuous function on M.

We claim that (1.21) remains valid for all $u \in B(M)$. In fact, let u_i be a bounded montone sequence of functions convergent pointwise to u. Then as J_n is a non-negative bounded operators, $J_n u_i$ is also a bounded monotone sequence of continuous functions. On each compact domain, the DeGiorgi-Nash-Moser theory tells us that we have a Hölder estimate of $J_n u_i$ in terms of $\| J_n u_i \|$ and $\| u_i \|$ so that $\lim_{i \to \infty} J_n u_i$ is also Hölder continuous. In particular, $J_n u_i$ converges uniformly on compact sets of M. Similarly, we can prove that $\lim_{i \to \infty} J_n u_i$ has locally finite Dirichlet integral so that $\lim_{i \to \infty} J_n u_i$ is the (unique) bounded solution of $x - (L\,x/n) = u$. Therefore, on compact sets, $J_n u_i$ converges uniformly to $J_n u = J_n(\lim_{i \to \infty} u_i)$. As a result, $\lim_{i \to \infty} T_t^n u_i(x) = T_t^n(\lim_{i \to \infty} u_i)(x)$, and (1.21) is valid when $u \in B(M)$.

Now the measure $p_t^n(x, \mu)$ on $\beta(M)$ has a subsequence $\{ p_t^{n_i}(x, \mu) \}$ weak* convergent to a Borel measure $p_t(x, \mu)$ defined on $\beta(M)$ so that for all bounded continuous function φ on $\beta(M)$:

(1.22)
$$\lim_{i \to \infty} \int_{\beta(M)} p_t^{n_i}(x, \mu)\, \varphi = \int_{\beta(M)} p_t(x, \mu)\, \varphi.$$

Since we know that $T_t^{n_i}$ is given by (1.21) and $T_t^{n_i}$ converges strongly to $T_t u$ for $u \in E$, we conclude that $\int_{\beta(M)} p_t(x, \mu)\, u$ is independent of the choice of the sequence $\{n_i\}$ as far as $u \in E$. In particular, the restriction of $p_t(x, \mu)$ to M is independent of the choice of $\{n_i\}$.

We claim that the measures $p_t(x, \mu)$ restricted to M are absolutely continuous. It suffices to prove the following statement. Let $\{u_i\}$ be a monotonic decreasing sequence of non-negative continuous functions with support lying in a fixed compact set. Suppose $\{u_i\}$ converges pointwisely to the characteristic function of a set of measure zero. Then for each $x \in M$, $\lim_{i \to \infty} T_t(u_i)(x) = 0$.

We shall prove that $\{J_n u_i\}$ tends to zero uniformly on compact sets. Let K be any smooth domain and $u_{i,K}$ be the solution of the equation $x-(1/n)\,L\,x = u_i$ which vanishes on ∂K. Then multiplying the equation by $u_{i,K}$ and integrating, we can dominate the L^2-norm of $u_{i,K}$ by the L^2-norm of u_i. Taking limit, we have dominated the L^2-norm of $J_n u_i$ by the L^2-norm of u_i. From equation (1.15), we see that the L^2-norm of $T_t^n u_i$ is also dominated by the L^2-norm of u_i. Therefore, for each i, some subsequence of $\{T_t^n u_i\}$ converges weakly to a function whose L^2-norm is dominated by the L^2-norm of u_i. Since $\lim_{n\to\infty} T_t^n u_i = T_t u_i$ strongly, the L^2-norm of $T_t u_i$ is dominated by the L^2-norm of u_i.

Now we can localize our argument. Fix a coordinate neighbor $N(x)$ of x. Then as $T_t u_i$ verifies the equation $(\partial/\partial t)(T_t u_i) = L(T_t u_i)$, the estimate of [9], p. 113, then shows that $T_t u_i$ is estimated from above by the L^2-norm of $T_t u_i$. Hence we can estimate $|T_t u_i(x)|$ by the L^2-norm of u_i which decreases to zero. This proves our claim and the fact that $p_t(x, \mu)$ are absolutely continuous. We can now write $p_t(x, \mu)$ as $p(t, x, y)\, dy$.

We assert that the measures $p_t(x, \mu)$ are in fact supported in M. To see this, we notice that the functions r_a are Lipschitz functions with compact support in M. Therefore they belong to our Banach space E and we can apply the Hille-Yosida theory to the function

$$n \int_0^\infty e^{-ns} T_s r_a \, ds,$$

to conclude that

$$J_n r_a = n \int_0^\infty e^{-ns} T_s r_a \, ds.$$

[This follows from the uniqueness of the solutions of $x-(1/n)\,L\,x = u$.] Hence, by (1.10), we have

$$n \int_0^\infty e^{-ns} T_s r_a(x)\, ds \geqq r_a(x) - n^{-1} C_a \tag{1.23}$$

for all $x \in M$.

Since

$$\| r_a \| = 1, \qquad \lim_{a\to\infty} r_a(x) = 1 \qquad \text{and} \qquad \lim_{a\to\infty} C_a = 0;$$

we have

$$1 \geqq n \int_0^\infty e^{-ns} \int_M p_s(x, \mu)\; ds \geqq \lim_{a\to\infty} (r_a(x) - n^{-1} C_a) = 1. \tag{1.24}$$

Therefore, the inequalities in (1.33) become equalities and we have

$$\int_{M} p_s(x, \mu) = 1 \tag{1.25}$$

for almost all s.

In order to prove that (1.25) is valid for all s, we notice that from the equation $T_t T_s \varphi = T_{t+s} \varphi$ for all φ with compact support, we can derive the following:

$$\int_{\beta(M)} p_t(x, \mu(y)) p(s, y, z) = p(t+s, x, z) \tag{1.26}$$

for almost every $z \in M$.

For all $s > 0$, we can pick $t > 0$ so that $\int_{M} p(t+s, x, z)\, dz = 1$. Integrating (1.26), we obtain:

$$1 \geqq \int_{\beta(M)} p_t(x, \mu(y)) \left(\int_{M} p(s, y, z)\, dz \right) = 1 \tag{1.27}$$

which implies $\int_{M} p(s, y, z)\, dz = 1$ for almost every y except on a set of zero $p_t(x, \mu(y))$ measure.

However, as an easy consequence of Moser's Harnack inequality [9], one can show that on compact sets, $p_t x, \mu(y))$ dominates $p_{t'}(x, \mu(y))$ for $t' < t$. As t is arbitrary, we see that $\int_{M} p(s, y, z)\, dz = 1$ except on a set of zero $p_t(x, \mu(y))$ measure for any t.

Now for all $t > 0$, we can pick $s_1 > 0$ and $s_2 > 0$ so that $t = s_1 + s_2$ and $\int_{M} p(s_1, x, y)\, dy = 1$. Then integrating (1.26) with respect to z, we have

$$\int_{M} p(t, x, z)\, dz = \int_{M} p(s_1, x, y)\, dy \int_{M} p(s_2, y, z)\, dz = 1. \tag{1.28}$$

Hence, we have proved that (1.25) is true for all $s > 0$ and $x \in M$. In particular, we have proved that for all $\varphi \in E$,

$$T_t \varphi(x) = \int_{M} p(t, x, y) \varphi(y)\, dy. \tag{1.29}$$

We can simply extend our semi-group T_t to define on $B(M)$ by the formula (1.29). The equations (1.25) and (1.26) guarantees that T_t is indeed a semi-group.

Following Dynkin, we restrict our semi-group to the following space:

$$B = \{ u \in B(M) : \lim_{s \to 0} T_s u(x) = u(x) \text{ for all } x \in M \}. \tag{1.30}$$

Then, clearly for all $t \geqq 0$, $T_t B \subseteq B$ and we claim that for all $u \in B$,

$$J_n u = n \int_{0}^{\infty} e^{-ns} T_s u \in B \qquad \text{for} \quad n > 0.$$

In fact, the formula is definitely valid for $u \in \mathrm{E}$. On the other hand, we have observed that if $\{u_i\}$ is a bounded monotone sequence, then $\lim\limits_{i \to \infty} \mathrm{J}_n u_i(x) = \mathrm{J}_n \, (\lim\limits_{i \to \infty} u_i)(x)$. Hence, the above formula must be valid for general u (note that for all u, $\mathrm{J}_n u \in \mathrm{B}$).

Now it follows from the well-known theory (*see* for example [4]) that for $n > 0$:

$$\mathrm{J}_n \mathrm{B} = \left\{ u \in \mathrm{B} : \frac{1}{h}(\mathrm{T}_h u(x) - u(x)) \text{ converges boundedly to a function in B for } h \to 0 \right\} \tag{1.31}$$

and for all $u \in \mathrm{J}_n \mathrm{B}$,

$$\lim_{h \to 0} \frac{1}{h}(\mathrm{T}_h u(x) - u(x)) = n u(x) - n \mathrm{J}_n^{-1} u(x). \tag{1.32}$$

If $u \in \mathrm{B}$ and $\mathrm{L} u$ is bounded, then $\mathrm{J}_n[u - \mathrm{L}(u/n)] = u$. Therefore, the domain of L is a subset of $\mathrm{J}_n \mathrm{B}$. On the other hand, by definition of J_n, $\mathrm{J}_n \mathrm{B}$ is a subset of the domain of L. Combining this with (1.31) and (1.32), we see that the domain of L (restricted to B) is

$$\mathrm{D}(\mathrm{L}) = \left\{ u \in \mathrm{B} : \frac{1}{h}(\mathrm{T}_h u(x) - u(x)) \text{ converges boundedly to a function in B for } h \to 0 \right\} \tag{1.33}$$

and for $u \in \mathrm{D}(\mathrm{L})$:

$$\mathrm{L} u(x) = \lim_{h \to 0} h^{-1}(\mathrm{T}_h u(x) - u(x)). \tag{1.34}$$

Finally, we can state the main theorem of this section.

THEOREM 1. — *Let* L *be an elliptic operator defined on a manifold* M *so that either hypothesis* A, B, C *or hypothesis* A′, B, C *are satisfied. Then we can find a system of transition probability measures defined on* M *so that* L *is the generator of the corresponding semi-group* T_t *in the sense defined in* (1.33) *and* (1.34). *Furthermore, if* E *is the closure of the space of bounded functions in the domain of* L *with respect to the supremium norm, then* $\mathrm{T}_t \mid \mathrm{E}$ *is a semi-group in the sense of Hille-Yosida (when we consider* E *as a Banach space with the supremium norm) whose generator is given by* L *restricted to the space of bounded functions* u *so that both* $\mathrm{L} u$ *and* $\mathrm{L}^2 u$ *are bounded.*

2. Application

Suppose the Ricci curvature of M is bounded from below by a constant $(m-1)\mathrm{K}$ where m is the dimension of M. Let x_0 be a fixed point in M and r be the distance functions of M from x_0. Let M_{K} be a complete simply connected manifold with constant sectional curvature K and $\bar{r}$ be the distance function of M_{K} from a fixed point $\bar{x}$. Then it is well-known (*cf.* [1]) that within the cut locus of M at x_0, we have

$$\mathrm{L} r \leqq \bar{\mathrm{L}} \bar{r}, \tag{2.1}$$

where $\bar{\mathrm{L}}$ is the Laplace operator of M_{K}.

On the other hand, in the appendix of [8], we have shown that (2.1) remains valid in the distribution sense even if we consider r as a function defined globally. Hence we have the following:

THEOREM 2. — *Let* M *be a* m*-dimensional complete Riemannian manifold with Ricci curvature bounded from below by a constant* $(m-1)\,\mathrm{K}$. *Then the Laplace operator* L *of* M *is the generator of a semi-group* T_t *(defined by transition probabilities) in the following sense: Let* u *be a bounded continuous function. Then* $\mathrm{L}\,u$ *is bounded in the distribution sense iff* $t^{-1}(\mathrm{T}_t\,u(x)-u(x))$ *converges boundedly when* t *tends to zero. In this case* $\mathrm{L}\,u = \lim_{t\to 0} t^{-1}(\mathrm{T}_t\,u-u)$.

Finally, we want to show that the kernels $p(t, x, y)$ are small at infinity by proving the following:

THEOREM 4. — *Let* M *be a complete Riemannian manifold with Ricci curvature bounded from below by a constant. Let* T_t *be the (Brownian) semi-group associated to the Laplacian of* M *and* $\mathrm{C}_0(\mathrm{M})$ *be the space of continuous functions on* M *which vanish at infinity. Then* $\mathrm{T}_t(\mathrm{C}_0(\mathrm{M}) \subseteq \mathrm{C}_0(\mathrm{M})$ *for all* $t \geqq 0$.

Proof. — Clearly, we need only to prove that $\mathrm{T}_t\,u \in \mathrm{C}_0(\mathrm{M})$ whenever u is a non-trivial non-negative continuous function with compact support.

Let the support of u be a subset of $\mathrm{B}(x_0, \mathrm{R})$, the ball of radius R and center x_0, then we claim that there is a positive constant c depending only on the lower bound of the Ricci curvature such that

$$\mathrm{J}_n u(x) \leqq \frac{c}{n}\|u\|(d(x, x_0)-\mathrm{R})^{-1}, \tag{2.8}$$

when the distance $d(x, x_0)$ is greater than $\mathrm{R}+1$.

By the computation in [8], we know that if r is the distance function from x, then $\mathrm{L}\,r$, in the distribution sense, is bounded from above by $(m-1)/r$ plus a constant depending only on the lower bound of the Ricci curvature of M. Here $m = \dim \mathrm{M}$. Let σ be a non-decreasing smooth function define on the real line so that $\sigma(t) = t$ for $t \geqq 1/2$, $\sigma(0) = 0$ and $\sigma'(t) = 0$ for t near the origin. Then $\mathrm{L}(\sigma \circ r)$, in the distribution sense, is bounded from above by a constant depending only on the lower bound of the Ricci curvature.

Consider the function
$$\mathrm{J}_n u(y) - \frac{2\|\mathrm{J}_n u\|\,\sigma \circ r(y)}{d(x, x_0)-\mathrm{R}}$$
defined on M. It is negative when $r(y)$ is not less than $d(x, x_0)-\mathrm{R} \geqq 1$ and non-negative when $y = x$. Hence, it must achieve a maximum at a point y_0 in the interior of the ball $\mathrm{B}(x, d(x, x_0)-\mathrm{R})$.

Therefore at y_0,

$$n(\mathrm{J}_n u(y_0)-u(y_0)) = \mathrm{L}\mathrm{J}_n u(y_0) \leqq \frac{2\|\mathrm{J}_n u\|\,\mathrm{L}(\sigma \circ r)(y_0)}{d(x, x_0)-\mathrm{R}} \tag{2.9}$$

and

$$\begin{aligned} &\sup\left\{\mathrm{J}_n u(y) - \frac{2\|\mathrm{J}_n u\|\,\sigma \circ r(y)}{d(x, x_0)-\mathrm{R}} \;\middle|\; y \in \mathrm{B}(x, d(x, x_0)-\mathrm{R})\right\} \\ &\qquad \leqq u(y_0) + \frac{2\|\mathrm{J}_n u\|\,\mathrm{L}(\sigma \circ r)(y_0)}{n(d(x, x_0)-\mathrm{R})} = \frac{2\|\mathrm{J}_n u\|\,\mathrm{L}(\sigma \circ r)(y_0)}{n(d(x, x_0)-\mathrm{R})}. \end{aligned} \tag{2.10}$$

(Note that y_0 is not in the support of u.)

It follows from (2.10) that

$$J_n u(x) = J_n u(x) - \frac{2\|J_n u\| \sigma \circ r(x)}{d(x, x_0) - R} \leqq \frac{2\|J_n u\| L(\sigma \circ r)(y_0)}{n(d(x, x_0) - R)}. \tag{2.11}$$

Inequality (2.8) follows readily from (2.11).

Let us now prove $T_s u \in C_0(M)$ for all $s \geqq 0$. Suppose this is not true. Then for some s_0, we can find a sequence $\{x_i\}$ divergent to infinity such that

$$T_{s_0} u(x_i) \geqq \delta > 0 \tag{2.12}$$

for some δ independent of i.

On the other hand, if we set $S_i = \{ s \geqq 0 \mid T_s u(x_i) \geqq \delta/2 \}$, then it follows from (2.8) that

$$n \int_{S_i} e^{-ns} \leqq \frac{2c}{n\delta} \|u\| (d(x_i, x_0) - R)^{-1}. \tag{2.13}$$

Hence, if S_i^c is the complement of S_i in $[0, \infty)$, we have

$$\lim_{i \to \infty} n \int_{S_i^c} e^{-ns} = 1. \tag{2.14}$$

In particular, we can find a sequence $\{s_j\}$ such that $s_j \in S_j^c$ for all j and $s_j \to s_0$. By the strong continuity of T_s, we have a contradiction and $T_s u \in C_0(M)$ for all s.

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